Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(00)00061-7
J. Appl. Maths Mechs, Vol. 64, No. 3, pp. 387–398, 2000

0021-8928/00/$—see front matter

AN ASYMPTOTIC METHOD FOR SOLVING NON-STATIONARY DYNAMIC CONTACT PROBLEMS FOR AN ACOUSTICAL LAYER†

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(*Received* 15 *October* 1999)

An asymptotic method is proposed for solving non-stationary dynamic contact problems in elasticity theory and acoustics for the case when the half-thickness of the punch exceeds the layer thickness. The method is demonstrated by solving anti-plane non-stationary dynamic contact problems concerning the displacement by a rigid punch of an elastic layer, such problems are essentially the acoustic case of problems in elasticity theory. The problems are reduced to solving an integral equation of the first kind for the Laplace transforms of the unknown contact stresses. The zero term of the asymptotic solution of the integral equation is constructed as the superposition of solutions of the two corresponding Wiener–Hopf integral equations minus the solution of the corresponding integral equation over the entire axis [1]. The symbol of the kernel of the integral equation is represented in a special form which enables the solution of the Wiener–Hopf integral equation to be reduced to the solution of an integral equation of the second kind for the Laplace–Fourier transform of the unknown contact stresses. The solution of integral equations of the second kind is constructed by successive approximations. After Laplace inversion of the zero term of the asymptotic solution of the integral equation, the asymptotic solution of the problems under consideration is determined. Formulae are presented that relate the force acting on the punch to the displacement of the punch. A law of motion is obtained for a massive punch on an elastic layer for the case when an initial velocity was communicated to the punch at the initial time.

1. INTEGRAL EQUATIONS

Contact problems concerning a rigid punch penetrating an acoustic strip reduce to solving an integral equation of the first kind:

$$\int_{-1}^{1} \varphi^L(\xi, p) k\left(\frac{\xi - x}{\Lambda}, p\right) d\xi = 2\pi f^L(p), \quad |x| \leq 1 \tag{1.1}$$

where the unknown function $\varphi^L(x,p)$ is the Laplace transform of the distribution of the contact stresses under the punch and $f^L(p)$ is a known function related to the law of motion of the punch in an acoustic medium. The kernel of the equation has the form

$$k(t, p) = \int_\Gamma K(u, p) e^{iut} du \tag{1.2}$$

where $K(u, p)$ is the symbol of the kernel and, Γ is an integration contour in the complex plane $u = \sigma + i\tau$. Non-stationary dynamic contact problems of elasticity theory concerning the anti-plane displacement of an elastic layer by a rigid punch may be reduced to the form (1.1).

We will consider two classical non-stationary dynamic contact problems (henceforth called NSDCPs) concerning the anti-plane displacement by a rigid punch of width $2a(|x| \leq a)$ of an elastic layer of width h: the lower side of the layer ($y = 0$) is rigidly fixed to a non-deformable base (Problem A); the lower side of the elastic layer is stress-free (Problem B). Up to the initial time ($t = 0$) the layer is at rest.

Mixed boundary conditions for problems A and B are given by the following formulae ($y = h$, $t > 0$) [2]

$$w(x, h, t) = \varepsilon(t), \quad |x| \leq a; \quad w'_y(x, h, t) = 0, \quad a < |x| < \infty \tag{1.3}$$

where $\varepsilon(t)$ is the law of motion of the punch and the prime denotes partial differentiation.

†*Prikl. Mat. Mekh.* Vol. 64, No. 3, pp. 403–415, 2000.

The following condition holds at the lower side of the layer ($y = 0, t > 0$)

$$w(x,\ 0,\ t) = 0, \quad |x| < \infty \quad \text{(problem A)} \tag{1.4}$$

$$w'_y(x,\ 0,\ t) = 0, \quad |x| < \infty \quad \text{(problem B)} \tag{1.5}$$

where $w(x, y, t)$ is a function representing the displacements of the elastic layer along the Oz, axis, which satisfies the equation

$$\Delta w = c^{-2}\partial^2 w / \partial t^2 \tag{1.6}$$

(Δ is the Laplacian and c is the velocity of sound in the elastic medium). The function $w(x, y, t)$ and its partial derivatives tend to zero as $|x|,\ |y| \to \infty$.

To reduce problems (1.3)–(1.6) to the solution of an integral equation, we apply integral transformations, taking Laplace transformations with respect to time t and Fourier transformations with respect to the x coordinate [3]

$$w^L(x,\ y,\ p) = \int_0^\infty w(x,\ y,\ t)e^{-pt}dt \tag{1.7}$$

$$w^{LF}(\alpha,\ y,\ p) = \int_{-\infty}^\infty w^L(x,\ y,\ p)e^{i\alpha x}dx \tag{1.8}$$

These transformations reduce problems A and B to solving integral equation (1.1) of the first kind with kernel (1.2). The symbol of the kernel of the integral equation is given by the formula

$$K(u,\ p) = \sigma^{-1}\,\mathrm{th}(\gamma\Lambda^{-1}\sigma) \quad \text{(problem A)} \tag{1.9}$$

$$K(u,\ p) = \sigma^{-1}\,\mathrm{cth}(\gamma\Lambda^{-1}\sigma),\ \ \sigma = \sqrt{u^2+1} \quad \text{(problem B)} \tag{1.10}$$

In formulae (1.1), (1.2), (1.9) and (1.10)

$$\Lambda = \frac{c}{ap},\quad \gamma = \frac{h}{a},\quad f^L(p) = \frac{G}{a}\varepsilon^L(p)$$

where $\varepsilon^L(p)$ is the Laplace transform of the function $\varepsilon(t)$, G is the shear modulus of the elastic layer and Γ is an integration contour in the complex plane $u = \sigma + i\tau$, making an angle $-\arg p$ with t the real axis ($\tau = 0$). To compute the root in (1.10), we take the branch for which $\sqrt{1} = 1$.

The functions $K(u, p)$ given by formulae (1.9) and (1.10) are even, meromorphic in the complex plane $u = \sigma + i\tau$ and have there an even number of zeros and poles, whose values may be determined by elementary means. At large values of $|u|$ both functions have the asymptotic behaviour

$$K(u,\ p) = |u|^{-1} + O(|u|^{-3}),\quad |u| \to \infty$$

and for small u

$$K(u,\ p) = A + O(u^2),\quad u \to 0;\quad A = \begin{cases} \mathrm{th}(\gamma\Lambda^{-1}) & \text{for problem A} \\ \mathrm{cth}(\gamma\Lambda^{-1}) & \text{for problem B} \end{cases}$$

The functions $K(u,p)$ (1.9), (1.10) may be expressed in a special form, as series of exponential functions [4]:

$$K(u,\ p) = \sigma^{-1}\left(1 - 2\sum_{n=1}^\infty d_n \exp(-b_n\sigma)\right),\quad \mathrm{Re}\,\sigma > 0 \tag{1.11}$$

$$d_n = \begin{cases} (-1)^{n+1} & \text{for problem A (1.9)} \\ -1 & \text{for problem B (1.10)} \end{cases},\quad b_n = 2n\gamma\Lambda^{-1} \tag{1.12}$$

The factor σ^{-1} on the right-hand side of (1.11) is the symbol of the kernel of the integral equation corresponding to the anti-plane NSDCP of the displacement of an elastic half-space by a rigid punch [5]. This is easily seen by letting h, $\gamma \to \infty$ in formulae (1.9)–(1.11). In the general case, the symbol of the kernel $K(u, p)$ of the integral equation of the NSDCP for an acoustic layer may be expressed in special form as a series

$$K(u, p) = K(u) - 2\sum_{n=1}^{\infty} q_n(u)\exp(-b_n\sigma)K(u), \quad \operatorname{Re}\sigma > 0 \tag{1.13}$$

When representation (1.11) is used for $K(u, p)$, the kernel of integral equation (1.2) may be written as a series

$$k(t, p) = 2K_0(t) - 4\sum_{n=1}^{\infty} d_n K_0\left(\sqrt{b_n^2 + t^2}\right)$$

where $K_0(t)$ is the MacDonald function and the quantity d_n is defined by formula (1.12)

2. APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION

The zero term of the asymptotic solution of Eq. (1.1) for large p and h < a is conveniently constructed as the superposition of the solutions of the following integral equations [1, 2, 6, 7]

$$\int_{-1}^{\infty} \varphi_+^L(\xi, p)k\left(\frac{\xi - x}{\Lambda}, p\right)d\xi = 2\pi f^L(p), \quad -1 \leqslant x < \infty \tag{2.1}$$

$$\int_{-\infty}^{1} \varphi_-^L(\xi, p)k\left(\frac{\xi - x}{\Lambda}, p\right)d\xi = 2\pi f^L(p), \quad -\infty < x < 1 \tag{2.2}$$

$$\int_{-\infty}^{\infty} \varphi_\infty^L(\xi, p)k\left(\frac{\xi - x}{\Lambda}, p\right)d\xi = 2\pi f^L(p) \quad -\infty < x < \infty \tag{2.3}$$

according to the formula

$$\varphi^L(x, p) = \varphi_+^L\left(\frac{1 + x}{\Lambda}, p\right) + \varphi_-^L\left(\frac{1 - x}{\Lambda}, p\right) - \varphi_\infty^L\left(\frac{x}{\Lambda}, p\right) \tag{2.4}$$

The kernels $k(t, p)$ of Eqs (2.1)–(2.3) are identical with (1.2), where the integration contour Γ is deformed in the complex plane $u = \sigma + i\tau$ and coincides with the real axis ($\tau = 0, -\infty < \sigma < \infty$).

We will make the following change of variables in (2.1) and (2.2) (a plus sign for (2.1) and a minus sign for (2.2)

$$\pm\xi = \Lambda\xi' - 1, \quad \pm x = \Lambda x' - 1, \quad \pm d\xi = \Lambda d\xi'$$

As a result, the two equations (2.1) and (2.2) become Wiener–Hopf integral equations [1, 7]

$$\int_{0}^{\infty} \varphi_+^L(\xi, p)k(\xi - x)d\xi = 2\pi\frac{f^L(p)}{\Lambda}, \quad 0 \leqslant x < \infty \tag{2.5}$$

Applying to Eq. (2.3) the change of variables

$$\xi = \Lambda\xi', \quad x = \Lambda x', \quad d\xi = \Lambda d\xi'$$

we obtain an integral equation of convolution type over the entire axis [8]

$$\int_{-\infty}^{\infty} \varphi_\infty^L(\xi, p)k(\xi - x)d\xi = 2\pi\frac{f^L(p)}{\Lambda}, \quad -\infty < x < \infty \tag{2.6}$$

Constructing a solution of integral equation (2.5) for $\varphi_+^L(x, p)$ we thereby also determine $\varphi_-^L(x, p)$, since in the case under consideration $\varphi_+^L(x, p) = \varphi_-^L(x, p)$.

Following the general scheme for solving Wiener–Hopf integral equations [7, 8] to determine $\varphi_+^L(x, p)$, at the first stage of the solution procedure we extend the definition of Eq. (2.5) to the entire real axis and then, using the Fourier integral transformation (1.8), we reduce the problem to solving the functional equation

$$K(u,\ p)\Phi_+(u,\ p) = -\frac{1}{iu}\frac{f^L(p)}{\Lambda} + \frac{1}{2\pi}E_-(u,\ p) \tag{2.7}$$

which holds in a certain strip $\tau_- < \mathrm{Im}(u) < \tau_+$ of the complex plane $u = \sigma + i\tau$, where we have introduced the notation

$$\Phi_+(u,\ p) = \int_0^\infty \varphi_+(\xi,\ p)\exp(iu\xi)d\xi \tag{2.8}$$

$$E_-(u,\ p) = \int_{-\infty}^0 e(\xi)\exp(iu\xi)d\xi, \quad e(x,\ p) = \int_{-\infty}^0 \varphi_+^L(\xi,\ p)k(\xi - x)d\xi \quad -\infty < x < 0$$

Here, the function $\Phi_+(u, p)$ is regular in the upper half-plane ($\mathrm{Im}(u) > \tau_-$, $-1 \leqslant \tau_- \leqslant 0$) and $E_-(u, p)$ is regular in the lower half-plane ($\mathrm{Im}(u) < \tau_+$, $0 < \tau_+ \leqslant 1$) and $E_-(u, p)$. The functions $K(u, p)$, defined by (1.9) and (1.10), are regular at least in the strip $|\mathrm{Im}(u)| < 1$.

Let us express the function $K(u, p)$ in Eq. (2.7) as a series (1.13), where $K(u)$ is the symbol of the kernel of the integral equation of the corresponding NSDCP for the displacement of an elastic half-space by a punch [5]

$$K(u) = \sigma^{-1} \tag{2.9}$$

and d_n and b_n are defined by formulae (1.12).

Substituting series (1.13) for $K(u, p)$ into Eq. (2.7) and factorizing the function $K(u)$ [7], which in this case may be done by elementary means,

$$K(u) = K_+(u)K_-(u), \quad K_+(u) = K_-(-u) = (1 - iu)^{-1/2} \tag{2.10}$$

and dividing the resulting equation by $K_-(u)$, we obtain a functional equation

$$K_+(u)\Phi_+(u,\ p) = -\frac{f^L(p)}{\Lambda}g(u) + 2\sum_{n=1}^\infty d_n M^n(u,\ p) + \frac{1}{2\pi}\frac{E_-(u,\ p)}{K_-(u)} \tag{2.11}$$

$$g(u) = \frac{1}{iuK_-(u)}, \quad M^n(u,\ p) = K_+(u)\Phi_+(u,\ p)\exp(-b_n\sigma)$$

The functions $g(u)$ and $M^n(u, p)$ may be expressed as the sum of two functions, one regular in the upper half-plane ($\mathrm{Im}(u) > \tau_-$) and the other in the lower half-plane ($\mathrm{Im}(u) < \tau_+$) of the complex plane $u = \sigma + i\tau$

$$g(u) = g_+(u) + g_-(u) \tag{2.12}$$

$$M^n(u) = M_+^n(u) + M_-^n(u) \tag{2.13}$$

This may be done, for example, by using a general theorem [7], whence we obtain

$$g_+(u) = \frac{1}{iuK_-(0)}, \quad g_-(u) = \frac{1}{iu}\left[\frac{1}{K_-(u)} - \frac{1}{K_-(0)}\right] \tag{2.14}$$

$$M_+^n(u,\ p) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} K_+(\zeta)\Phi_+(\zeta,\ p)\exp(-b_n\sigma)\frac{d\zeta}{\zeta - u}, \quad \tau_- < c < \tau_+$$

$$M_-^n(u,\ p) = M^n(u,\ p) - M_+^n(u,\ p) \tag{2.15}$$

Substituting expressions (2.12) and (2.13) into (2.11) and collecting functions regular in the upper half-plane ($\mathrm{Im}(u) > \tau_-$) on the left of the equality and functions regular in the lower half-plane ($\mathrm{Im}(u) < \tau_+$) on the right of the same equality, we obtain an equality whose left- and right-hand sides in combination define a certain function in the complex plane $u = \sigma + i\tau$. On the assumption that the functions $\Phi_+(u,p)$ and $E_-(u,p)/K_-(u)$ decrease as $|u| \to \infty$, it can be shown that the left- and right-hand sides of the equality decrease at infinity. Then the function they define decreases as $|u| \to \infty$. By Liouville's theorem [7, 8], such a function must vanish identically in the complex plane $u = \sigma + i\tau$. In that case, the last equality yields two relationships defining $\Phi_+(u,p)$ and $E_-(u,p)$

$$K_+(u)\Phi_+(u,\ p)+\frac{f^L(p)}{\Lambda}g_+(u)-2\sum_{n=1}^{\infty} d_n M_+^n(u,\ p)=0 \tag{2.16}$$

$$-\frac{f^L(p)}{\Lambda}g_-(u)K_-(u)+2K_-(u)\sum_{n=1}^{\infty} d_n M_-^n(u,\ p)+E_-(u,\ p)=0 \tag{2.17}$$

Determining the required function $\Phi_+(u,p)$ from (2.16) and developing the formula thus obtained, we obtain an integral equation of the second kind for $\Phi_+(u,p)$

$$\Phi_+(u,\ p)=-\frac{f^L(p)}{\Lambda}\frac{g_+(u)}{K_+(u)}+\frac{1}{\pi i K_+(u)}\sum_{n=1}^{\infty} d_n\int_{\Gamma} N_n(\zeta,\ p)\frac{\Phi_+(\zeta,\ p)}{\zeta-u}d\zeta \tag{2.18}$$

$$N_n(u,\ p)=K_+(u)\exp(-b_n\sigma).$$

The contour Γ is situated in the regularity strip of the functional equation $\tau_- < \mathrm{Im}(u) < \tau_+$. Under those conditions, if $u \in \Gamma$, then (2.18) is a singular integral equation of the second kind, and the integral on its right-hand side may be understood in the sense of the Cauchy principal value [7].

Integral equations similar to (2.18) and approaches to their solution have been considered before [7]. Here we will propose a different method for solving Eq. (2.18), which takes the specific properties of its kernel into account.

The singular integral in (2.18) ($u \in \Gamma$), for $N_n(\zeta,p)\Phi_+(\zeta,p) \in L_q(\Gamma)$ ($1 < q < \infty$), is a bounded linear mapping of the function space $L_q(p)$ onto itself for any $q(1 < q < \infty)$ [10]. A solution of Eq. (2.18) may therefore be sought by the method of successive approximations.

When estimating the integral operator in (2.18) as $|u| \to \infty$, which is necessary in order to determine the structure of the method of successive approximations, the estimate $K_+(u) = O(|u|^{-1/2}(|u| \to \infty)$ is taken into account on the assumption that $\Phi_+(u) = O(|u|^{-1/2}(|u| \to \infty)$, and subsequently the method of steepest descent [11] is used to investigate the operator, yielding the following inequality ($|u| \to \infty$)

$$\left|\frac{1}{2\pi}\int_{\Gamma} N_n(\zeta,\ p)\frac{\Phi_+(\zeta,\ p)}{\zeta-u}d\zeta\right| < \frac{|c_0(p)|}{|u|}\exp(-b_n) \tag{2.19}$$

where $c_0(p)$ is a constant, which depends on p and the integration contour Γ is situated in the regularity strip $\tau_- < \mathrm{Im}(u) < \tau_+$ and may coincide with the real axis. The singular point $\zeta = u$ is circumvented from below if u $u \in \Gamma$.

To solve integral equation (2.18) by successive approximations, we propose the following iterative scheme

$$\Phi_+^{m+1}(u,\ p)=-\frac{f^L(p)}{\Lambda}\frac{g_+(u)}{K_+(u)}+\frac{1}{\pi i K_+(u)}\sum_{n=1}^{\infty} d_n\int_{\Gamma} N_n(\zeta,\ p)\frac{\Phi^m(\zeta,\ p)}{\zeta-u}d\zeta \tag{2.20}$$

$$m=0,\ 1,\ 2,\ \ldots$$

in which the zeroth approximation $\Phi_+^0(u,\ p)$ is taken to be the known function on the right of Eq. (2.20)

$$\Phi_+^0(u,\ p)=-\frac{f^L(p)}{\Lambda}\frac{g_+(u)}{K_+(u)} \tag{2.21}$$

which, in turn, is the Laplace–Fourier transform of the solution of the corresponding NSDCP of the displacement of an elastic half-space by a punch [5].

When implementing the iterative process (2.20) of the method of successive approximations, the computed quadratures are generally written as special functions and the structure of the process is difficult to determine.

The structure of the iterative process is determined using an asymptotic analysis of the integral operator in (2.20).

For $m = 0$ Eq. (2.20) has the special form

$$\Phi_+^1(u,\ p) = \Phi_+^0(u,\ p) + \Delta\Phi_+^0(u,\ p) \tag{2.22}$$

$$\Delta\Phi_+^0(u,\ p) = \frac{1}{K_+(u)} \sum_{n_1=1}^{\infty} d_{n_1} \int_{\Gamma_1} N_{n_1}(\zeta_1,\ p) \frac{\Phi_+^0(\zeta_1,\ p)}{\zeta_1 - u} d\zeta_1$$

Using inequality (2.19) in this case, we obtain the following estimate for $\Delta\Phi_+^0(u,p)$ as $|u| \to \infty$

$$|\Delta\Phi_+^0(u,\ p)| \leq \frac{|c_1(p)|}{|u|^{1/2}} \sum_{n=1}^{\infty} d_{n_1} \exp(-b_{n_1}) \tag{2.23}$$

The function $\Phi_+^1(u,p)$ contains an infinite number of terms, each involving (in the asymptotic sense) an exponential term $\exp(-b_{n_1})$ ($n_1 = 1, 2, \ldots$). These terms will be refined in subsequent steps of the successive-approximation process.

When $m = 1$, after substituting $\Phi_+^1(u,p)$ from (2.22) into Eq. (2.20), we get

$$\Phi_+^2(u,\ p) = \Phi_+^1(u,\ p) + \Delta\Phi_+^1(u,\ p) \tag{2.24}$$

$$\Delta\Phi_+^1(u,\ p) = \frac{1}{(\pi i)^2 K_+(u)} \sum_{n_2=1}^{\infty} d_{n_2} \sum_{n_1=1}^{\infty} d_{n_1} \times$$

$$\times \int_{\Gamma_2} \frac{\exp(-b_{n_2}\sigma(\zeta_2))}{\zeta_2 - u} d\zeta_2 \int_{\Gamma_1} N_{n_1}(\zeta_1,\ p) \frac{\Phi_+^0(\zeta_1,\ p)}{\zeta_1 - \zeta_2} d\zeta_1$$

One obtains the following estimate for $\Phi_+^1(u,p)$ as $|u| \to \infty$

$$|\Delta\Phi_+^1(u,\ p)| \leq \frac{|c_2(p)|}{|u|^{1/2}} \sum_{n_2=1}^{\infty} \sum_{n_1=1}^{\infty} \exp(-b_{n_1+n_2}) \tag{2.25}$$

indicating that one must add to the infinite sum of terms in $\Phi_+^1(u,p)$ the double sum of terms with exponentials $\exp(-b_{n_1} - b_{n_2})$ whose minimum exponent is $(b_{n_1} + b_{n_2}) = b_2$, since $b_{n_1} + b_{n_2} = b_{n_1+n_2}$ and $\inf(n_1 + n_2) = 2$. Hence it follows that the double sum $\Phi_+^1(u,p)$ does not contain a term with the exponential $\exp(-b_1)$ and, naturally, does not introduce a correction to the term in $\Phi_+^1(u,p)$ containing an exponential with the same exponent $-b_1$. All other terms of $\Phi_+^1(u,p)$ with exponentials $\exp(-b_2)$, $\exp(-b_3)$, ... receive corrections from the double sum $\Delta\Phi_+^1(u,p)$.

Thus, $\Phi_+^2(u,p)$ contains only one term (the first) with the exponential $\exp(-b_1)$ in the infinite sum, that does not receive a correction from the subsequent successive approximations. The function $\Phi_+^1(u,p)$, the second approximation to $\Phi_+(u,p)$, contains two terms with exponentials $\exp(-b_1)$ and $\exp(-b_2)$ that experience no changes in subsequent successive approximations.

For arbitrary m, the integration scheme (2.20) may be written in the form

$$\Phi_+^{m+1}(u,\ p) = \Phi_+^m(u,\ p) + \Delta\Phi^m(u,\ p),\quad m = 0,\ 1,\ 2, \ldots \tag{2.26}$$

$$\Delta\Phi_+^m(u,\ p) = \frac{1}{(\pi i)^{m+1} K_+(u)} \sum_{n_{m+1}=1}^{\infty} d_{n_{m+1}} \sum_{n_m=1}^{\infty} \cdots \sum_{n_1=1}^{\infty} d_{n_{m+1}} \int_{\Gamma_{m+1}} \frac{\exp(-l_{n_{m+1}}(\zeta_{m+1}))}{\zeta_{m+1} - u} d\zeta_{m+1} \times$$

$$\times \int_{\Gamma_m} \frac{\exp(-l_{n_m}(\zeta_m))}{\zeta_m - \zeta_{m+1}} d\zeta_m \cdots \int_{\Gamma_1} \frac{K_+(\zeta_1)\Phi_+^0(\zeta_1,\ p)}{\zeta_1 - \zeta_2} \exp(-l_{n_1}(\zeta_1)) d\zeta_1$$

$$l_{n_j}(u) = b_{n_j}\sigma(u),\quad \sigma(u) = (u^2 + 1)^{1/2}$$

where Γ_j are integration contours in the regularity strip $\tau_- < \mathrm{Im}(u) < \tau_+$ ($j = 1, 2, \ldots, m + 1$) and $\Phi_+^0(u,p)$ is given by formula (2.21).

The estimate

$$|\Delta\Phi_+^m(u,p)|\leq\frac{|c_m(p)|}{|u|^{1/2}}\sum_{n_{m+1}=1}^{\infty}\dots\sum_{n_m=1}^{\infty}\sum_{n_1=1}^{\infty}\exp(-b_{n_1+n_2+\dots n_{m+1}}),\quad |u|\to\infty \tag{2.27}$$

shows that the infinite sum representing $\Phi_+^{m+1}(u, p)$ begins with $m + 1$ terms that experience no changes in subsequent iterations, since only the first term of the $(m + 1)$-fold sum in $\Phi_+^m(u, p)$ with least exponent in the exponential $\exp(-b_{m+1})$ ($\inf(n_1 + n_2 + \dots + n_{m+1}) = m + 1$ and $b_{n_1} + b_{n_2} + \dots + b_{n_{m+1}} = b_{n_1+n_2+\dots+n_{m+1}}$) corrects the $(m + 1)$th term contained in $\Phi_+^m(u, p)$, as well as all subsequent terms in which the exponents of the exponentials have larger indices: $b_{m+1}, b_{m+3}, \dots$. The first $m + 1$ terms in $\Phi_+^{m+1}(u, p)$ constitute a mathematical description of $m + 1$ elastic waves reflected from the lower side of the layer through the layer to the bottom of the punch; to determine these terms, one has to perform m iterations in (2.20).

It is important to note that the approximate solutions $\Phi_+^j(u, p)$ ($j = 1, 2, \dots, m + 1$) obtained for integral equation (2.18) are of the same order of accuracy with respect to u as $\Phi_+^0(u, p)$. This is confirmed by estimates (2.13), (2.25) and (2.27).

To construct an approximate solution for the NSDCP under consideration, with one wave reflected from the lower side of the layer through the layer to the bottom of the punch, one has to take the solution $\Phi_+^1(u, p)$ (2.22) of integral equation (2.18). Substituting (2.21) into (2.22) and using the explicit expressions (2.10) for $K_\pm(u)$, we obtain

$$\Phi_+^1(u,p)=\Phi_+^0(u,p)+\sum_{n=1}^{\infty}d_n\exp(-b_n)\Phi_+^0(u,p)+$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}d_nb_n\Phi_+^0(u,p)\int_0^{\infty}\frac{K_1(\upsilon_n(\xi))\exp(iu\xi)}{\upsilon_n(\xi)}d\xi,\quad \upsilon_n(x)=\sqrt{b_n^2+x^2} \tag{2.28}$$

where $K_1(u)$ is the MacDonald function.

Taking inverse Fourier transformations of (2.21) and (2.28), we find the zeroth and first approximations of the solution of Eq. (2.5)

$$\varphi_\pm^{mL}(x,p)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_+^m(u,p)e^{-iux}du,\quad m=0,1,2,\dots \tag{2.29}$$

Implementation of (2.29) yields the following formulae [3, 4, 12]

$$\varphi_\pm^{0L}(x,p)=\frac{f^L(p)}{\Lambda}\left(\operatorname{erf}\sqrt{x}+\frac{\exp(-x)}{\sqrt{\pi x}}\right) \tag{2.30}$$

$$\varphi_\pm^{1L}(x,p)=\varphi_\pm^{0L}(x,p)+\sum_{n=1}^{\infty}d_n\exp(-b_n)\varphi_\pm^{0L}(u,p)+$$

$$+\frac{1}{2\pi}\sum_{n=1}^{\infty}d_nb_n\int_0^{x/b_n}\frac{K_1(b_n\upsilon(\xi))\varphi_\pm^{0L}(x-b_n\xi,p)}{\upsilon(\xi)}d\xi,\quad \upsilon(x)=\sqrt{1+x^2} \tag{2.31}$$

The zeroth approximation $\varphi^{0L}(x, p)$ does not contain elastic waves reflected from the lower side of the layer to the bottom of the punch (corresponding to the solution of the NSDCP for an elastic half-plane), as indicated by the superscript zero; the first approximation $\varphi^{1L}(x, p)$, contains one elastic wave reflected from the lower side of the layer, as indicated by the superscript 1.

Integral equation (2.6) for the NSDCP under consideration may be solved exactly using the Fourier integral transformation (1.8); the solution is

$$\varphi_\infty^L(x,p)=\frac{f^L(p)}{\Lambda}\left[1-2\sum_{n=1}^{\infty}g_n\exp(-b_n)\right],\quad g_n=\begin{cases}-1 & \text{for problem A}\\(-1)^{n+1} & \text{for problem B}\end{cases} \tag{2.32}$$

To solve Eq. (2.6), one can use the method of successive approximations proposed for solving integral equation (2.18). To that end, the solution of Eq. (2.6) is reduced by the use of Fourier integral transformation (1.8) to the solution of the functional equation

$$K(u,p)\Phi_\infty(u,p) = 2\pi\frac{f^L(p)}{\Lambda}\delta(u) \tag{2.33}$$

where $\delta(u)$ is the Dirac delta-function. The functions $K(u, p)$ for problems A and B are expressed in the special form (1.11) and substituted into (2.33); after dividing by $K(u)$, we obtain

$$\Phi_\infty(u,p) = 2\pi\frac{f^L(p)}{\Lambda}\frac{\delta(u)}{K(u)} + 2\sum_{n=1}^{\infty} d_n \exp(-b_n\sigma)\Phi_\infty(u,p)$$

The iterative process is organized in a scheme analogous to (2.20)

$$\Phi_\infty^{m+1}(u,p) = 2\pi\frac{f^L(p)}{\Lambda}\frac{\delta(u)}{K(u)} + 2\sum_{n=1}^{\infty} d_n \exp(-b_n\sigma)\Phi_\infty^m(u,\ p),\quad m=0,1,2,\ldots \tag{2.34}$$

$$\Phi_\infty^0(u,p) = 2\pi\frac{f^L(p)}{\Lambda}\frac{\delta(u)}{K(u)}$$

which may be written as a new scheme analogous to (2.26)

$$\Phi_\infty^{m+1}(u,p) = \Phi_\infty^m(u,\ p) + \Delta\Phi_\infty^m(u,\ p),\quad m=0,1,2,\ldots \tag{2.35}$$

$$\Delta\Phi_\infty^m(u,p) = 2\sum_{n_{m+1}=1}^{\infty} d_{n_{m+1}}\, 2\sum_{n_m=1}^{\infty} d_{n_m} \ldots 2\sum_{n_1=1}^{\infty} d_{n_1} \exp(-b_{n_1+n_2\ldots+n_{m+1}})\Phi_\infty^0(u,p)$$

Repeating the arguments described for implementing interative scheme (2.26), it is easy to establish, on the basis of (2.35), that for $m = 0$ scheme (2.34) gives

$$\Phi_\infty^1(u,p) = 2\pi\frac{f^L(p)}{\Lambda K(u)}\left(1 + 2\sum_{n=1}^{\infty} d_n \exp(-b_n\sigma)\right)\delta(u) \tag{2.36}$$

Using the properties of the delta-function in Eq. (2.36) and taking inverse Fourier transformations, we obtain the solution

$$\varphi_\infty^{1L}(x,p) = \frac{f^L(p)}{\Lambda}\left(1 + 2\sum_{n=1}^{\infty} d_n \exp(-b_n)\right) \tag{2.37}$$

in which the zeroth and first term are identical with the exact solution (2.32), while the function $\varphi_\infty^{m+1,L}(x,p)$ thus obtained will have its first $m + 2$ terms identical with the exact solution (2.32).

The zeroth term of the asymptotic solution of integral equation (1.1), containing the description of one wave (the first) reflected from the lower side through the layer to the bottom of the punch, is constructed by the formula

$$\varphi^{1L}(x,p) = \varphi_+^{1L}\left(\frac{1+x}{\Lambda},p\right) + \varphi_-^{1L}\left(\frac{1-x}{\Lambda},p\right) - \varphi_\infty^{1L}\left(\frac{x}{\Lambda},p\right) \tag{2.38}$$

where the superscript indicates the number of reflected waves contained in the solution. The function $\varphi_+^{1L}(x,p)$ in (2.38) is given by formula (2.31), and the function $\varphi_\infty^{1L}(x,p)$ by formula (2.37).

3. SOLUTION OF THE NSDCPS UNDER CONSIDERATION

In the previous section we determined an approximate solution for the integral equation of the NSDCP – the function $\varphi^{1L}(x, p)$ (2.38), which is the Laplace transform of the unknown contact stresses. To determine the solution of the NSDCPs formulated in Section 1, we need only evaluate the inverse Laplace transformation of $\varphi^{1L}(x,p)$ (2.38). This gives a solution of the problem in the from

$$\varphi^1(x,t) = \varphi_+^1\left(\frac{a(1+x)}{c},t\right) + \varphi_-^1\left(\frac{a(1-x)}{c},t\right) - \varphi_\infty^1\left(\frac{ax}{c},t\right) \tag{3.1}$$

where

$$\varphi^1_{\pm}(x,t)=\varphi^0_{+}(x,t)+\sum_{n=1}^{\infty} d_n\varphi^0_{+}(x,t-\beta_n)+\frac{2}{\pi}\sum_{n=1}^{\infty} d_n\frac{d}{dt}\int_0^{\xi_*}\frac{d\xi}{v^2(\xi)}\int_{t_*}^{t}\frac{\tau\varphi^0_{+}(x-\beta_n\xi,t-\tau)}{\sqrt{\tau^2-\beta_n^2v^2(\xi)}}d\tau \tag{3.2}$$

$$\varphi^0_{+}(u,t)=\frac{G}{c}\left[\frac{1}{\pi\sqrt{u}}\frac{d}{dt}\int_u^t \tau^{-1}\sqrt{\tau-u}\,\varepsilon(t-\tau)d\tau+\varepsilon'(t)-\varepsilon(\beta_n)\right] \tag{3.3}$$

$$\varphi^1_{\infty}(u,t)=\frac{G}{c}\left[\varepsilon'(t)+\varepsilon(0)+2\sum_{n=1}^{\infty} d_n(\varepsilon'(t-\beta_n)+\varepsilon(\beta_n))\right] \tag{3.4}$$

Here

$$t_*=\max\{\beta_n\sqrt{\xi^2+1},x-\beta_n\xi\}\quad \xi_*=\min\left\{\frac{\sqrt{t^2-\beta_n^2}}{\beta_n},\frac{a(1+x)}{\beta_n c}\right\},\quad \beta_n=2n\frac{h}{c}$$

and for the function $\varphi^0_{+}(x, t)$ outsider the sum in (3.2), $\beta_n = 0$; G is the shear modulus.

In the simplest case, when $\varepsilon(t) = w_0H(t)$, where $H(t)$ is the Heaviside function, the solution of the NSDCP has the simplest form (3.1), where

$$\varphi^0_{+}(u,t)=\frac{Gw_0}{c}\left[\frac{\sqrt{t-u}}{\pi t\sqrt{u}}+\delta(t)\right],\quad \varphi^1_{\infty}(u,t)=\frac{Gw_0}{c}\left[2\sum_{n=1}^{\infty} d_n\delta(t-\beta_n)+\delta(t)\right] \tag{3.5}$$

and $\varphi^1_{\pm}(x, t)$ is evaluated by formula (3.2)

Formulae (3.1)–(3.5) enable one to analyse the nature of the wave field of the stresses beneath the punch, including that formed in the period during which the first wave reflected from the lower side of the layer reaches the bottom of the punch. If $0 < t < \beta_1$ (before the arrival of a reflected wave), the field of contact stresses is identical with the field of contact stresses in the problem of a punch displacing an elastic half-space [5]. If $\beta_1 < t, < \beta_2$, the wave of stresses reflected from the lower side of the layer and reaching the bottom of the punch at time $t = \beta_1$ generates new waves of contact stresses, which spread from the edges of the punch as from sources with the velocity of sound c.

There are no singularities on the fronts of these waves at $\beta_1 < t < \beta_2$, as for $0 < t < \beta_1$; but the constant singularities of the stresses at the punch edges appear in the solution.

4. THE MAGNITUDE OF THE FORCE APPLIED TO THE PUNCH

The magnitude of the force applied to the punch and causing the punch to move according to the law $\varepsilon(t)$ is evaluated by the formula

$$T(t)=a\int_{-1}^{1}\varphi(x,t)dx \tag{4.1}$$

In the case under consideration, the Laplace transform of this force may be found using the formula

$$T^{1L}(p)=a\int_{-1}^{1}\varphi^{1L}(x,p)dx \tag{4.2}$$

(the function $\varphi^{1L}(x, p)$ is given by formula (2.38)). In this case formula (4.2) may be written in the form

$$T^{1L}(p)=T^{1L}_{+}(p)+T^{1L}_{-}(p)-T^{1L}_{\infty}(p) \tag{4.3}$$

$$T^{1L}_{\pm}(p)=a\int_{-1}^{1}\varphi^{1L}_{\pm}(x,p)dx,\quad T^{1L}_{\infty}(p)=a\int_{-1}^{1}\varphi^{1L}_{\infty}(x,p)dx$$

In our case, we have ($T_+^{1L}(p) = T_-^{1L}(p)$)

$$T_+^{1L}(p) = T_+^{0L}(\theta, p) + \sum_{n=1}^{\infty} d_n \exp(-b_n) T_+^{0L}(\theta, p) +$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty} d_n b_n \int_0^{a/(nh)} \frac{K_1'(b_n\upsilon(\xi)) T_+^{0L}(\theta - b_n\xi, p)}{\upsilon(\xi)} d\xi, \quad \theta = \frac{2}{\Lambda} = \frac{2ap}{c} \tag{4.4}$$

$$T_+^0(u,p) = a\frac{f^L(p)}{\Lambda}\left[\left(u+\frac{1}{2}\right)\operatorname{erf}\sqrt{u} + \sqrt{\frac{u}{\pi}}\exp(-u) + u\right]$$

$$T_\infty^1(p) = 2a\frac{f^L(p)}{\Lambda}\left[1 + 2\sum_{n=1}^{\infty} d_n \exp(-b_n)\right]$$

Taking an inverse Laplace transformation in (4.3), we obtain (($T_+^1(t) = T_-^1(t)$))

$$T^1(t) = 2T_+^1(t) - T_\infty^1(t) \tag{4.5}$$

$$T_+^1(t) = T_+^0(2a/c, t) + \sum_{n=1}^{\infty} d_n T_+^0(2a/c, t - \beta_n) +$$

$$+\frac{4}{\pi}\sum_{n=1}^{\infty} d_n \frac{d}{dt}\int_0^{\xi_*} \frac{d\xi}{\upsilon^2(\xi)} \int_{t_*}^{t} \frac{\tau T_+^0(2a/c - \beta_n\xi, t-\tau)}{\sqrt{\tau^2 - \beta_n^2\upsilon^2(\xi)}} d\tau$$

$$\beta_n = 2n\frac{h}{c}, \quad \xi_* = \frac{2a - ct}{nh}, \quad t_* = \beta_n\upsilon(\xi)$$

where

$$T_+^0(u,t) = G\left[\frac{1}{2}\varepsilon(t) + u(\varepsilon'(t) + \varepsilon(0)) - \frac{\sqrt{u}}{2\pi}S_1(u,t) + \frac{\sqrt{u}}{\pi}\frac{d}{dt}S_2(u,t) - \frac{u\sqrt{u}}{\pi}\frac{d}{dt}S_1(u,t)\right] \tag{4.6}$$

$$S_1(u,t) = \int_u^t \frac{\varepsilon(t-\tau)}{\tau\sqrt{\tau - u}} d\tau, \quad S_2(u,t) = \int_u^t \frac{\varepsilon(t-\tau)}{\tau\sqrt{\tau - u}} d\tau.$$

$$T_\infty^1(t) = \frac{2a}{c}G\left[\varepsilon'(t) + \varepsilon(0) + 2\sum_{n=1}^{\infty} d_n(\varepsilon'(t-\beta_n) + \varepsilon(\beta_n))\right]$$

In the simplest case, when $\varepsilon(t) = w_0 H(t)$, we have instead of (4.6)

$$T_+^0(u,t) = Gw_0\left[\frac{1}{2}H(t) - \frac{1}{\pi u}\operatorname{arctg}\sqrt{u(t-u)} + \frac{u\sqrt{u(t-u)}}{\pi(1 + u(t-u))} + u\delta(t)\right]$$

$$T_\infty^1(t) = \frac{2a}{c}Gw_0\left[2\sum_{n=1}^{\infty} d_n\delta(t-\beta_n) + \delta(t)\right]$$

Note that, as before, the infinite sum in (4.5) contains only one first reflected wave, described by the first term of the sum; the others are computed formally and have to be corrected. Analysis of the formula shows that the terms in (4.5) outside the sum constitute the force $T(t)$ of the corresponding NSDCP for a half-space ($0 < t < \beta_1$) [9]. The first term in (4.5) in the infinite sum corresponds to an algebraic increment to the force at $\beta_1 < t, < \beta_2$, due to the arrival of the first wave of stresses reflected from the lower side of the layer.

5. THE MOTION OF THE PUNCH

Let us determine the law of motion of the punch $\varepsilon(t)$, knowing its linear mass m and the velocity $\dot\varepsilon(0) = v_0$ at the initial instant of time ($t = 0$). In that case the equation of motion of the punch will be

$$m\ddot{\varepsilon}(t) = -T(t) \tag{5.1}$$

with initial conditions $\dot{\varepsilon}(0) = \nu_0$, $\varepsilon(0) = \varepsilon_0$ where ε_0 is the initial displacement of the punch, up to the time $t = 0$. The elastic force $T(t)$ resisting the motion of the punch, due to the contact stresses between the punch and the elastic layer, will be considered to be equal to the magnitude $T^1(t)$ for the initial time interval $0 < t < 4hc^{-1}$, as defined by formula (4.5).

Using the methods of the operational calculus [13] to solve Eq. (5.1), we obtain the following expression for the transform $\varepsilon^L(p)$ of the unknown function $\varepsilon(p)$

$$m\lfloor p^2\varepsilon^L(p) - \varepsilon(0)p - \dot{\varepsilon}(0)\rfloor = -T^{1L}(p) \tag{5.2}$$

where $T^{1L}(p)$ is given by formula (4.3). In the case under consideration

$$\varepsilon^L(p) = \frac{\varepsilon(0)p + \dot{\varepsilon}(0)}{mp^2 + GN(p)} \tag{5.3}$$

$$N(p) = 2m_+^0(\theta, p) + 2\sum_{n=1}^{\infty} d_n \exp(-b_n) m_+^0(\theta, p) - 2\sum_{n=1}^{\infty} d_n \exp(-b_n)\theta +$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty} d_n b_n \int_0^{a/(nh)} \frac{K_1(b_n \upsilon(\xi)) m_+^0(\theta - b_n\xi, p)}{\upsilon(\xi)} d\xi - \theta \tag{5.4}$$

$$m_+^0(\theta, p) = (\theta + ½)\operatorname{erf}\sqrt{\theta} + \sqrt{\gamma/\pi}\exp(-\theta) + \theta$$

To obtain an approximate solution $\varepsilon(t)$, we replace all functions in (5.3) and (5.4) by their asymptotic forms (p, $\theta \to \infty$).

$$m_+^0(\theta, p) = (\theta + ½ - \exp(-\theta)/(2\sqrt{\pi\theta}) + O(\theta^{-3/2}\exp(-\theta)), \quad \theta \to \infty$$

$$K_1(u) = \sqrt{\frac{\pi}{2u}}\exp(-u)\left(1 + O\left(\frac{1}{u}\right)\right), \quad u \to \infty$$

and use the method of steepest descent [11] to evaluate the integral in (5.4). We then obtain the approximate formula

$$\varepsilon^L(p) = \frac{\varepsilon(0)p + \dot{\varepsilon}(0)}{M(p)} + \frac{\varepsilon(0)p + \dot{\varepsilon}(0)}{M^2(p)} Q(p), \quad M(p) = mp^2 + \theta G + G$$

$$Q(p) = G\left[\frac{\exp(-\theta)}{\sqrt{\pi\theta}} - \sum_{n=1}^{\infty} d_n \exp(-b_n)\left(4\theta + 3 - 3\frac{\exp(-\theta)}{\sqrt{\pi\theta}}\right)\right] \tag{5.5}$$

For small t ($0 < t, < \beta_2$) we obtain

$$\varepsilon(t) = \varepsilon(0)(\dot{E}(t) + \dot{D}(t)) + \dot{\varepsilon}(0)(E(t) + D(t)) \tag{5.6}$$

where

$$D(t) = \int_{\beta n}^{t} \tau E(\tau) F(t - \tau) d\tau$$

$$F(t) = \frac{G}{\sqrt{\pi\tau_0}}\left[2\sqrt{t - \tau_0} - \sum_{n=1}^{\infty} d_n (4\tau_0 \delta(t - \beta_n) + 3H(t - \beta_n) - 6\sqrt{t - \tau_0 - \beta_n})\right]$$

$$E(t) = m^{-1}\exp(-\delta t) \times \begin{cases} \omega^{-1}\sin\omega t, & \omega = \sqrt{\delta_0 - \delta^2} \\ \omega^{-1}\operatorname{sh}\omega t, & \omega = \sqrt{\delta^2 - \delta_0} \\ t, & \omega = 0 \end{cases} \tag{5.7}$$

$$\delta = aG/(cm), \quad \delta_0 = G/m, \quad \tau_0 = 2a/c, \quad \beta_n = 2nh/c$$

Formula (5.6) for $\varepsilon(t)$ ($\varepsilon(0) = 0$) shows that, depending on the sign of the quantity $\Delta = a^2\rho - m$ (where ρ is the density of the layer material), the law of motion of the punch will be a decaying motion if $\Delta \geqslant 0$ and an oscillatory-attenuating motion if $\Delta < 0$. The first and second terms in the second formula of (5.7) under the summation sign ($n = 1$) indicate, in the case of Problem A, that the wave of stresses reflected from the lower (rigidly fixed) side of the layer, reaching the bottom of the punch, acts on the punch in a direction opposite to that of its preliminary motion ($d_1 = 1$), while in the case of Problem B that wave imparts to the punch an additional impulse in the direction of its original displacement ($d_1 = -1$).

The formulae obtained here hold up to the time $2a/c$ required for an elastic wave to proceed from one edge of the punch to the other. They are meaningful when $h < a$, when at least one elastic wave reflected from the lower side of the layer succeeds in reaching the bottom of the punch.

I wish to thank V. M. Aleksandrov for his interest.

This research was supported financially by the Russian Foundation for Basic Research (00-01-00-428a).

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Translated by D. L.